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Regularity of Weak Solutions to Elliptic Equations of Arbitrary Order

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It is well known that every weak solution (with boundary values 0) of a semilinear equation $Au + f(x, u) = g$ is a regular solution if f fulfils the growth condition (*) $|f(x, u)| \leqslant c\,|u|^{(n+2m)/(n-2m)-\epsilon}$. Here $2m$ is the order of A. In this paper we weaken this condition to $c\,|u|^{(n+2m)/(n-2m)+1} \geqslant f(x, u)u \geqslant -c\,|u|^{(n+2m)/(n-2m)+1-\epsilon}$. This requires a technique completely different from that which may be applied in case (*).

Introduction and Notation

In recent times considerable effort has been devoted to solving weakly semilinear elliptic equations of type

$$Au + f(x, u) = g, \tag{0.1}$$

where A is an elliptic operator of order $2m$, with divergence structure and bounded from below (see, e.g., [2]). The open set $\Omega \subset \mathbb{R}^n$, $n > 2m$, is assumed to be bounded and to have a smooth boundary. The function u is assigned boundary values 0.

In this paper we show that every weak solution of (0.1) is a regular solution if f fulfills the growth condition

$$c_1\,|u|^{(n+2m)/(n-2m)+1} \geqslant f(x, u)\,u \geqslant -c_2\,|u|^{(n+2m)/(n-2m)+1-\epsilon},$$
$$x \in \Omega, \quad |u| \geqslant 1, \tag{0.2}$$

with positive constants c_1, c_2. The number $\epsilon > 0$ may be arbitrarily small. Of course g must be a regular function. The growth condition on f from below in (0.2) seems to be the best possible one even in the case $m = 1$ (see [3, p. 285]). The growth condition on f from above is needed, since for higher-order equations no maximum principle such as that for second-order equations is abailable. This seems to be the best possible condition, since for the growth exponent $(n + 2m)/(n - 2m)$ the expression $f(u)$ is still in $H^{-m}(\Omega)$ if $u \in \mathring{H}^m(\Omega)$.

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We introduce the following notation. Let Ω be a bounded open set of $\mathbb{R}^n$. Then $H^{\nu,p}(\Omega)$, $\infty > p \geqslant 1$, is the Banach space of all real-valued $L^p(\Omega)$-functions with distributional derivatives up to the order ν which are elements of $L^p(\Omega)$. Its norm is denoted by $\|\cdot\|_{\nu,p}$. As usual $\mathring{H}^{\nu,p}$ is the completion of $C_0^\infty(\Omega)$ in the $\|\cdot\|_{\nu,p}$-norm. We set $H^\nu(\Omega) := H^{\nu,2}(\Omega)$, $\mathring{H}^\nu(\Omega) := \mathring{H}^{\nu,2}(\Omega)$. For functions f continuous in $\bar{\Omega}$ we set

$$w(f)(r) := \sup_{\substack{x,y\in\bar{\Omega}\\ |x-y|\leqslant r}} |f(x) - f(y)|.$$

The set Ω or $\partial\Omega$ is said to be of class C^k, $k \in \mathbb{N}$, if $\partial\Omega$ locally admits a representation

$$x_l = \phi(x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_n)$$

with a C^k-function ϕ, and if Ω is locally on one side of $\partial\Omega$. Finally we define D^α by

$$D^\alpha = \prod_{j=1}^{n} \left(\frac{1}{i}\frac{\partial}{\partial x_j}\right)^{\alpha_j}$$

if $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j \in \mathbb{N}$.

1. The Main Theorem and Its Proof

Let $m \in \mathbb{N}$ be a fixed number >0. Let Ω be a bounded open set of $\mathbb{R}^n$ with a boundary $\partial\Omega$ of class $4m$. For every pair (α, β) of multi-indices of $\mathbb{R}^n$ with $|\alpha|, |\beta| \leqslant m$ let the functions

$$A_{\alpha\beta} \in C^m(\bar{\Omega})$$

be given with $(1/i)^{|\alpha+|+|\beta|} A_{\alpha\beta} \in \mathbb{R}$ and the property

$$M^{-1} |\xi|^{2m} \leqslant \sum_{\substack{|\alpha|=m\\ |\beta|=m}} A_{\alpha\beta}(x)\, \xi^\alpha \xi^\beta \leqslant M |\xi|^{2m}, \qquad x \in \bar{\Omega}, \quad \xi \in \mathbb{R}^n,$$

for an $M > 0$. For an element $u \in H^{2m,p}$ with a $p > 1$ we write

$$Au = \sum_{\substack{|\alpha|\leqslant m\\ |\beta|\leqslant m}} D^\alpha(A_{\alpha\beta}(x)\, D^\beta u).$$

Let a function

$$f\colon \Omega \times \mathbb{R} \to \mathbb{R}$$

be given with the properties: f is measurable with respect to x, for all $u \in \mathbb{R}$, and f is continuous with respect to u, a.e. $x \in \Omega$,

$$\begin{aligned} | f(x, u)| &\leqslant \tilde{M} \mid u \mid^{(n+2m)/(n-2m)} + \tilde{K}, \\ f(x, u)\, u &\geqslant -c \mid u \mid^{(n+2m)/(n-2m)+1-\epsilon} - c'u^2. \end{aligned}$$

Here $\tilde{M}$, $\tilde{K}$, c, c', ϵ are positive constants with $1 > \epsilon > 0$.

An element $u \in \mathring{H}^m(\Omega)$ is a weak solution of the equation $Au + f(x, u) = g$, $g \in L^{2n/(n+2m)}(\Omega)$, if

$$\sum_{\substack{|\alpha| \leqslant m \\ |\beta| \leqslant m}} (A_{\alpha\beta}(x)\, D^\alpha u, D^\beta v) + (f(x, u), v) = (g, v)$$

for all $v \in \mathring{H}^m(\Omega)$. Observe that $f(x, u) \cdot v$ is in $L^1(\Omega)$ because of our growth condition. Without loss of generality we assume that $n > 2m$.[1] We can now formulate our *theorem* on the regularity of weak solutions to semilinear elliptic equations, namely,

THEOREM 1.1. *Let $g \in L^\infty(\Omega)$. Let u be a weak solution of $Au + f(x, u) = g$. Then $u \in \bigcap_{p>1} H^{2m,p}(\Omega)$.*

For the proof we need

THEOREM 1.2. *Let $\tilde{g} \in L^q(\Omega)$ for a $q > 2n/(n + 2m)$. Let $H: \Omega \times \mathbb{R} \to \mathbb{R}$ be a measurable mapping with the properties:*

H is a Caratheodory function, $H(x, 0) = 0$,

$H(x, u) \geqslant H(x, v)$, $x \in \Omega$, $u, v \in \mathbb{R}$, $u \geqslant v$,

$| H(x, u)| \leqslant \tilde{M} \mid u \mid^{1+4m/(n-2m)} + \tilde{K}$, $x \in \Omega$, $u \in \mathbb{R}$,

with positive constants $\tilde{M}$, $\tilde{K}$. Let $(Au, u) \geqslant c(n, m, M, \| A_{\alpha\beta} \|_{L^\infty(\Omega)}, w(A_{\alpha\beta}), \Omega) \| u \|^1_{m,2}$, $u \in H^{2m}(\Omega) \cap \mathring{H}^m(\Omega)$, with a positive constant $c(n,\ldots)$. Then every problem

$$P_\sigma \quad \begin{cases} Au + H(x, u) = \sigma\tilde{g}, \\ D^\alpha u \mid \partial\Omega = 0, \qquad | \alpha | \leqslant m - 1, \quad 0 \leqslant \sigma \leqslant 1, \end{cases}$$

has a unique solution $u_\sigma \in H^{2m,p}(\Omega) \cap \mathring{H}^{m,q}(\Omega)$ with

$$\begin{aligned} \| u_\sigma \|_{2m,q} &\leqslant c(n, m, \Omega, q, M, \| A_{\alpha\beta} \|_{L^\infty(\Omega)}, \| \nabla^m A_{\alpha\beta} \|_{L^\infty(\Omega)}, \tilde{M}, \tilde{K}, C), \\ C &\geqslant \| \tilde{g} \|_{L^q(\Omega)} . \end{aligned} \tag{1.2}$$

Proof. We introduce two sets Σ, Σ^*, defined as follows. Σ is the set of all σ,

[1] If $n < 2m$ every weak solution of $Au + f(x, u) = g$ is regular if g is regular, and no growth condition on f is needed; if $n = 2m$, f has to fulfill only a condition $| f(x, u)| \leqslant c \mid u \mid^q$, q arbitrary. This follows from a Sobolev inequality and the linear theory.

$0 \leqslant \sigma \leqslant 1$, for which P_σ has a solution $u_\sigma \in H^{2m,q}(\Omega)$. Σ^* is the set of all σ, $0 \leqslant \sigma \leqslant 1$, with the properties:

1. $[0, \sigma] \subset \Sigma$,

2. $\| u_\tau \|_{2m,q} \leqslant c(\sigma, n, m, \Omega, q, M, \| A_{\alpha\beta} \|_{L^\infty(\Omega)}, \| \nabla^m A_{\alpha\beta} \|_{L^\infty(\Omega)}, \tilde{M}, \tilde{K}, C)$, $C \geqslant \| \tilde{g} \|_{L^q(\Omega)}$, $0 \leqslant \tau \leqslant \sigma$.

Our aim is to prove that $\Sigma = \Sigma^* = [0, 1]$. First we remark that $0 \in \Sigma, \Sigma^*$. Let $\sigma_1 \in \Sigma$, $\sigma_2 \in \Sigma$. By scalar multiplication of $A(u_{\sigma_2} - u_{\sigma_1}) + H(x, u_{\sigma_2}) - H(x, u_{\sigma_1}) = (\sigma_2 - \sigma_1)\, \tilde{g}$ with $u_{\sigma_2} - u_{\sigma_1}$ we get

$$\| u_{\sigma_2} - u_{\sigma_1} \|_{m,2} \leqslant c(n, m, \Omega, M, \| A_{\alpha\beta} \|_{L^\infty(\Omega)}, w(A_{\alpha\beta}), C) \cdot | \sigma_2 - \sigma_1 |, \quad (1.3)$$

since H is monotonically increasing with respect to u. Observe that $H(x, u)\, v$ is integrable over Ω if $u, v \in H^{2m,q}(\Omega)$. On applying well-known estimates for linear equations, the triangle inequality to $| u_{\sigma_2} |^{1+4m/(n-2m)}$, and a well-known Sobolev inequality [3, p, 45], we get

$$\begin{aligned}
&\| u_{\sigma_2} - u_{\sigma_1} \|_{2m,q} \\
&\quad \leqslant c(n, m, \Omega, M, q, \| \nabla^m A_{\alpha\beta} \|_{L^\infty(\Omega)}, \| A_{\alpha\beta} \|_{L^\infty(\Omega)}, C) \\
&\qquad \cdot (\tilde{M} \| \, | u_{\sigma_2} |^{1+4m/(n-2m)} \|_{L^q(\Omega)} + \tilde{M} \| \, | u_{\sigma_1} |^{1+4m/(n-2m)} \|_{L^q(\Omega)} + \tilde{K}) \\
&\quad \leqslant c(n, m, \Omega, M, q, \| A_{\alpha\beta} \|_{C^m(\bar{\Omega})}, C) \\
&\qquad \cdot (\tilde{M} \| \, | u_{\sigma_2} - u_{\sigma_1} |^{1+4m/(n-2m)} \|_{L^q(\Omega)} + \tilde{M} \| \, | u_{\sigma_1} |^{1+4m/(n-2m)} \|_{L^q(\Omega)} + \tilde{K}) \\
&\quad \leqslant c(n, m, \Omega, M, q, \| A_{\alpha\beta} \|_{C^m(\bar{\Omega})}, C) \\
&\qquad \cdot (\tilde{M} \| u_{\sigma_2} - u_{\sigma_1} \|_{2m,q} \| u_{\sigma_2} - u_{\sigma_1} \|_{L^p(\Omega)}^{4m/(n-2m)} + \tilde{M} \| \, | u_{\sigma_1} |^{1+4m/(n-2m)} \|_{L^q(\Omega)} + \tilde{K})
\end{aligned}$$

with $p = 2n/(n - 2m)$. Using (1.3) we see that $u_{\sigma_2} - u_{\sigma_1}$ fulfills the a priori estimate

$$\| u_{\sigma_2} - u_{\sigma_1} \|_{2m,q} \leqslant c(\sigma_1, m, n, q, \Omega, \tilde{M}, \tilde{K}, M, \| A_{\alpha\beta} \|_{L^\infty(\Omega)}, \| \nabla^m A_{\alpha\beta} \|_{L^\infty(\Omega)}, C)$$

if $| \sigma_2 - \sigma_1 | \leqslant \delta = \delta(n, m, q, \Omega, \tilde{M}, M, \| A_{\alpha\beta} \|_{C^m(\bar{\Omega})}, C)$. Here δ does not depend on σ_1. This shows that u_{σ_2} can be estimated a priori in the desired way, since $\sigma_1 \in \Sigma^*$, and thus u_{σ_2}, satisfies an a priori estimate (1.2). Thus we see that $[0, \sigma_1 + \delta] \cap \{\sigma \mid [0, \sigma] \subset \Sigma\} \subset \Sigma^*$. Let

$$\begin{aligned}
H_\nu(x, u) &= H(x, \nu), & u &\geqslant \nu, \\
&= H(x, u), & -\nu &< u < \nu, \\
&= H(x, -\nu), & u &\leqslant -\nu,
\end{aligned}$$

$\nu \in \mathbb{N}$, $x \in \Omega$, $u \in \mathbb{R}$. H is monotonically increasing with respect to u, and it can be easily proved that every problem

$$P_\sigma^\nu \quad \begin{cases} Au + H_\nu(x, u) = \sigma\tilde{g}, \\ D^\alpha u \mid \partial\Omega = 0, \qquad |\alpha| \leqslant m - 1, \end{cases}$$

has a unique solution $u_{\nu,\sigma} \in H^{2m,q}(\Omega) \cap \mathring{H}^{m,q}(\Omega)$, $0 \leqslant \sigma \leqslant \sigma_1 + \delta$. As we just proved, $u_{\nu,\sigma}$ satisfies an a priori estimate (1.2), with the constant independent of ν. Letting ν tend to $+\infty$, we see that

$$u_{\nu,\sigma} \rightharpoonup u_\sigma \qquad \text{in } H^{2m,q}(\Omega),$$
$$H_\nu(x, u_{\nu,\sigma}) \rightharpoonup H(x, u_\sigma) \qquad \text{in } L^q(\Omega)$$

(the latter is proved by an application of Egorov's theorem). Thus $[0, \sigma_1 + \delta] \cap \Sigma = [0, \sigma_1 + \delta]$. Since δ did not depend on σ_1, we have proved that $[0, 1] = \Sigma = \Sigma^*$. Thus the proof of our *theorem* is finished.

Now we prove our main *theorem*, formulated at the beginning of this section.

Proof of Theorem 1.1. Because of our growth condition on f we have

$$u(f(x, u) + c\,|\,u\,|^q u + c'u) \geqslant 0, \qquad x \in \Omega, \quad u \in \mathbb{R}, \quad q = \frac{n + 2m}{n - 2m} - \epsilon - 1. \tag{1.4}$$

Instead of

$$Au + f(x, u) = h$$

we consider the equation

$$\tilde{A}u + f(x, u) - c_0 u = h$$

with $\tilde{A} = A + c_0$. Here $c_0 > 0$ is chosen so large that Garding's inequality

$$(\tilde{A}u, u) \geqslant c(n, m, \Omega, \|A_{\alpha\beta}\|_{L^\infty(\Omega)}, w(A_{\alpha\beta}), M)\,\|u\|_{m,2}^2, \qquad u \in H^{2m}(\Omega) \cap \mathring{H}^m(\Omega),$$

holds with a positive constant $c(n, m, \ldots,)$. Setting $\tilde{f}(x, u) = f(x, u) - c_0 u$ we have because of (1.4):

$$u(\tilde{f}(x, u) + c\,|\,u\,|^q\, u + c'u + c_0 u) \geqslant 0, \qquad x \in \Omega, \quad u \in \mathbb{R}. \tag{1.5}$$

We set

$$\begin{aligned} G(x, \tilde{u}) = {} & \frac{\tilde{f}(x, u(x)) + c\,|\,u(x)|^q\, u(x) + c'u(x) + c_0 u(x)}{\tilde{M}\,|\,u(x)|^{4m/(n-2m)}\, u(x) + c\,|\,u(x)|^q\, u(x) + c'u(x) + c_0 u(x) + \tilde{K}u(x)} \\ & \cdot (\tilde{M}\,|\,\tilde{u}\,|^{4m/(n-2m)}\tilde{u} + c\,|\,\tilde{u}\,|^q\tilde{u} + c\,\tilde{u} + c_0\tilde{u} + \tilde{K}\tilde{u}), \\ & \qquad x \in \Omega, \quad u(x) \neq 0, \quad \tilde{u} \in \mathbb{R}, \\ = {} & 0 \qquad \text{otherwise}. \end{aligned}$$

Here u is the weak solution under consideration. Instead of our original equation we deal with

$$\begin{aligned}\tilde{A}\tilde{u} + G(x, \tilde{u}) &= h + 2c_0 u + c'u + c\,|\,u\,|^q\, u, \\ &= \tilde{g}.\end{aligned} \tag{1.6}$$

As is evident, G fulfills the assumptions of Theorem 1.2 with respect to x, $\tilde{u}$. Since

$$\tilde{g} \in L^{p_0}(\Omega), \qquad p_0 = \frac{2n}{n - 2m}\Big/ q > \frac{2n}{n + 2m},$$

Eq. (1.6) has a unique solution $\tilde{u} \in H^{2m, p_0}(\Omega) \cap \mathring{H}^{m, p_0}(\Omega)$. Consequently

$$\tilde{g} \in L^{p_1}(\Omega), \qquad p_1 = \frac{p_0 n}{n - 2p_0 m}\Big/ q > p_0 .$$

In general we get that $\tilde{g} \in L^{p_\nu}(\Omega)$,

$$p_\nu = \frac{p_{\nu-1} n}{(n - 2p_{\nu-1} m) q},$$

and

$$\frac{p_\nu}{p_{\nu-1}} = \frac{p_{\nu-1}(n - 2p_{\nu-2} m)}{p_{\nu-2}(n - 2p_{\nu-1} m)} \geqslant \frac{p_1}{p_0} > 1.$$

From this it follows that after finitely many steps we arrive at

$$\tilde{g} \in L^\infty(\Omega).$$

Thus

$$\tilde{u} \in \bigcap_{p>1} H^{2m, p}(\Omega).$$

Since according to Sobolev $G(x, v)\, w$ is integrable for $v, w \in H^m(\Omega)$, Eq. (1.6) has a unique weak solution, namely, $\tilde{u}$. Evidently u solves (1.6) weakly. Thus our theorem is proved.

References

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